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# The hydrodynamic interaction of two small freely-moving spheres in a linear flow field

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Two rigid spheres of radii $a$ and $b$ are immersed in infinite fluid whose velocity at infinity is a linear function of position. No external force or couple acts on the spheres, and the effect of inertia forces on the motion of the fluid and the spheres is neglected. The purpose of the paper is to provide a systematic and explicit description of those aspects of the interaction between the two spheres that are relevant in a calculation of the mean stress in a suspension of spherical particles subjected to bulk deformation. The most relevant aspects are the relative velocity of the two sphere centres ($\mathbf{V}$) and the force dipole strengths of the two spheres ($S'_{ij}, S''_{ij}$), as functions of the vector $\mathbf{r}$ separating the two centres.

It is shown that $\mathbf{V}$, $S'_{ij}$ and $S''_{ij}$ depend linearly on the rate of strain at infinity and can be represented in terms of several scalar parameters which are functions of $r/a$ and $b/a$ alone. These scalar functions provide a framework for the expression of the many results previously obtained for particular linear ambient flows or for particular values of $\mathbf{r}/a$ or of $b/a$. Some new results are established for the asymptotic forms of the functions both for $r/(a+b) \gg 1$ and for values of $r-(a+b)$ small compared with $a$ and $b$. A reasonably complete numerical description of the interaction of two rigid spheres of equal size is assembled, the main deficiency being accurate values of the scalar functions describing the force dipole strength of a sphere in the intermediate range of sphere separations.

In the case of steady simple shearing motion at infinity, some of the trajectories of one sphere centre relative to another are closed, a fact which has consequences for the rheological problem. These closed forms are described analytically, and also numerically in the case $b/a = 1$.

## 1. Introduction

The velocity distribution in the fluid near an immersed body is affected by the presence of a second body in the fluid unless it is far away. There is a consequent influence on the distribution of fluid stress at the surface of the first body, and so on its translational and rotational motion. Such hydrodynamic interaction effects are of direct interest in considerations of impact, coalescence and migration of small particles of different kinds immersed in fluid. In the case of suspended particles on which no external force or couple acts, hydrodynamic interaction occurs when the suspension as a whole is set in motion and may then affect the rheological properties of the suspension.

If a large number of particles are immersed in fluid with random positions, and with a mean distance between nearest neighbours which is large compared with the linear dimensions of a particle, the most important hydrodynamic interactions will be those between the (relatively few) *pairs of particles* which happen to be close to each other, since groups of three or more adjoining particles are even scarcer. This gives special significance to considerations of the interaction of just two particles otherwise alone in a large expanse of fluid. The effect of such pair-interactions on the mean settling speed of small spherical particles falling through fluid under gravity has recently been examined (Batchelor 1972); and in a paper which is a companion to the present one (Batchelor & Green 1972, to be referred to hereafter as paper II), we examine the effect of pair-interactions on the mean stress in a suspension of small force-free spherical particles which is subjected to a prescribed bulk deforming motion. Both these investigations concern statistically homogeneous suspensions in which the volume fraction of the particles ($c$) is small compared with unity, and consideration of the effect of pair-interactions yields results correct to the order of a power of $c$ which is one higher than those obtained by neglecting all interaction effects.

Our purpose in this paper is to provide information about the interaction of two force-free couple-free rigid spheres immersed in a moving fluid, with a view to its use in paper II for the determination of the mean stress, correct to the order $c^2$, in a moving suspension of many such particles. The velocity of the fluid far from the two spheres is taken to be a linear function of position of general form, and the flow near the spheres is dominated by viscous stresses. An explicit solution for the whole flow field in analytical form would be very complicated, and perhaps not usable, and it is desirable therefore to focus the inquiry on the features that are relevant to the rheological problem, especially the velocities of the two spheres and the force dipole strengths of the spheres. We shall give expressions for these relevant quantities, the forms of which are independent of the nature of the linear ambient flow field and of the ratio of the radii of the two spheres. These general expressions, which contain some unknown scalar functions, provide a framework for the expression of detailed results obtained for particular linear ambient flow fields and radius ratios by previous authors (Lin, Lee & Sather 1970 in particular). We shall also obtain some new approximate analytical results for the interaction of two spheres both when they are far apart and when they are very close together, and fit these into the same framework. A reasonably complete description of the interaction is obtainable in the case of two rigid spheres of equal size, and, although the values of the scalar functions are not yet known accurately in some parts of the range of values of the distance between the two sphere centres, it proves to be possible in paper II to obtain the desired estimate of the mean stress in a dilute suspension of rigid spherical particles of uniform size correct to the order $c^2$. More work is needed on the problem of two spheres of different size in a linear ambient flow before it will be possible to calculate the mean stress in a suspension of spheres of non-uniform size.

In most previous mathematical investigations of low-Reynolds-number flow involving two rigid spheres, or a rigid sphere and a plane rigid boundary (effectively a second sphere of infinite radius), the spheres have been assumed to be moving with specified velocities, or moving under the action of specified forces or couples, through fluid at rest at infinity (e.g. Stimson & Jeffery 1926; Goldman, Cox & Brenner 1966, 1967*a*; O'Neill & Stewartson 1967; Cooley & O'Neill 1968, 1969; Davis 1969). The analytical methods used in these investigations are generally similar to those needed for the range of problems considered here, in which the two rigid spheres, or one sphere in the presence of a plane boundary, are force-free and couple-free and are immersed in fluid with a specified linear velocity variation far from the spheres; and a number of particular cases have been studied (Goldman, Cox & Brenner 1967*b*; Wakiya, Darabaner & Mason 1967; O'Neill 1968; Lin *et al.* 1970; Goren 1970; Goren & O'Neill 1971; Wakiya 1971). The results obtained for all such particular cases can be incorporated into the general representation of the interaction of two force-free and couple-free spheres that we now describe.†

## 2. General expressions for the sphere velocities and force dipole strengths

We consider two rigid spheres of radii $a$ and $b$ on which no external force or couple acts and which are immersed in incompressible fluid of viscosity $\mu$ with no other boundaries present. The spheres are of such small size that the Reynolds number of the fluid motion is small and inertia forces can be neglected. The ambient flow field (that is, the flow field in the absence of the two spheres) has velocity $\mathbf{U}(\mathbf{x}, t)$ which is assumed to be a linear function of position and can therefore be characterized instantaneously by a uniform rate-of-strain tensor

$$E_{ij} = \frac{1}{2}\left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j}\right)$$

and a rigid-body rotation with angular velocity

$$\mathbf{\Omega} = \tfrac{1}{2}\nabla \times \mathbf{U}.$$

The instantaneous position of the centre of the sphere of radius $a$ is $\mathbf{x}_0$ and that of the sphere of radius $b$ will be denoted by $\mathbf{x}_0 + \mathbf{r}$.

The formal hydrodynamic problem for the velocity $\mathbf{u}(\mathbf{x})$ and pressure $p(\mathbf{x})$ in the fluid is governed by the Stokes equations of motion

$$\nabla p = \mu \nabla^2 \mathbf{u}, \quad \nabla . \mathbf{u} = 0.$$

The outer boundary condition is

$$u_i(\mathbf{x}) \sim E_{ij}x_j + \epsilon_{ijk}\Omega_j x_k \quad \text{as} \quad |\mathbf{x}| \to \infty,$$

† In a paper published after this work had been submitted, Brenner & O'Neill (1972) have provided a similar, and more comprehensive, general framework for the expression of results concerning the forces and couples acting on two spheres moving with prescribed translational and rotational velocities in fluid whose velocity far from the spheres is a linear function of position. In the present paper we consider only the case of two rigid spheres on which no external force or couple acts, but we go further with this case by giving numerical results and by including consideration of the force dipole strengths of the spheres.

and the no-slip condition must be satisfied at the surfaces of the two spheres, $|\mathbf{x}-\mathbf{x}_0| = a$ and $|\mathbf{x}-\mathbf{x}_0-\mathbf{r}| = b$. The translational and rotational velocities of the two spheres are determined by the conditions that there is zero resultant force and couple exerted on each sphere by the hydrodynamic stress at the surfaces.

We wish in particular to determine certain parameters of this flow field involving two spheres, viz.

(1) the translational velocity of the centre of the sphere of radius $b$ relative to that of the sphere of radius $a$, to be denoted by $\mathbf{V}$;

(2) the angular velocities of the spheres of radii $a$ and $b$, to be denoted by $\mathbf{\Gamma}'$ and $\mathbf{\Gamma}''$ respectively; and

(3) the force dipole strengths of the two spheres, to be denoted by $S'_{ij}$ and $S''_{ij}$ respectively. (The force dipole strength $S_{ij}$ for a rigid body with surface $A_0$ and unit outward normal $\mathbf{n}$ is defined as

$$S_{ij} = \int_{A_0} (\sigma_{ik} x_j - \tfrac{1}{3}\delta_{ij}\sigma_{lk} x_l)\, n_k\, dA, \tag{2.1}$$

where $\sigma_{ij}$ is the hydrodynamic stress at position $\mathbf{x}$. The origin for the position vector in the integrand in (2.1) is arbitrary when the body is force-free. $S_{ij}$ is a symmetrical tensor, in the absence of an externally applied couple to the body, and by definition has zero trace. It may be shown (Batchelor 1967, § 4.11) that, if there is an outer boundary $A_1$ to the fluid at which the velocity is specified as $x_j \partial U_i/\partial x_j$, the direct contribution to the rate of dissipation of energy due to the presence of the body within $A_1$ is $E_{ij} S_{ij}$; this is one aspect of the connexion with the bulk dynamics which makes the force dipole strength of interest in the rheological problem considered in paper II.)

The quantities $\mathbf{V}, \mathbf{\Gamma}', \mathbf{\Gamma}'', S'_{ij}/\mu$ and $S''_{ij}/\mu$ are functions only of $\mathbf{r}$, $a$, $b$, $E_{ij}$ and $\mathbf{\Omega}$. The only instantaneous consequence of the rotation of the fluid at infinity is to superimpose a uniform angular velocity $\mathbf{\Omega}$ on the whole system. And relative to axes rotating with angular velocity $\mathbf{\Omega}$, the governing equations and boundary conditions are linear and homogeneous in the fluid velocity and pressure and $E_{ij}$. Consequently, each of the five quantities $\mathbf{V}-\mathbf{\Omega}\times\mathbf{r}$, $\mathbf{\Gamma}'-\mathbf{\Omega}$, $\mathbf{\Gamma}''-\mathbf{\Omega}$, $S'_{ij}$, $S''_{ij}$ must be linear and homogeneous in $E_{ij}$.

Thus with no loss of generality we may write for the polar vector $\mathbf{V}$

$$V_i(\mathbf{r}) = \epsilon_{ijk}\Omega_j r_k + r_j E_{ij} - \left\{A\frac{r_i r_j}{r^2} + B\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right\} r_k E_{jk}, \tag{2.2}$$

where $r = |\mathbf{r}|$ and the dimensionless scalar quantities $A$ and $B$ are functions only of the non-dimensional distance $r/a$ and the radius ratio $b/a$. The precise form of (2.2) has been chosen so as to make the radial and circumferential components of $\mathbf{V}$ depend only on $A$ and $B$ respectively. Likewise for the axial vectors $\mathbf{\Gamma}'$ and $\mathbf{\Gamma}''$ we have

$$\Gamma_i'^{\,\text{or}\,''} = \Omega_i + C'^{\,\text{or}\,''}\epsilon_{ijk} E_{kl}\frac{r_j r_l}{r^2}, \tag{2.3}$$

where $C'$ and $C''$ are dimensionless functions of $r/a$ and $b/a$ only. It follows from the definition of these scalar functions that

$$A(b/a) \equiv A(a/b), \quad B(b/a) \equiv B(a/b), \quad C'(b/a) \equiv C''(a/b).$$

When the two spheres are far apart, each sphere moves approximately as if it were alone in infinite fluid, that is,

$$A \to 0, \quad B \to 0, \quad C' \to 0, \quad C'' \to 0 \quad \text{as} \quad r/(a+b) \to \infty.$$

Similarly, we may express the force dipole strength of one sphere, which is a symmetrical tensor with zero trace, in the form

$$S'_{ij} = \tfrac{20}{3}\pi a^3 \mu \left\{ E_{ij}(1+K') + E_{kl}\left(\frac{r_i r_k \delta_{jl} + r_j r_k \delta_{il}}{r^2} - \frac{r_k r_l}{r^2}\tfrac{2}{3}\delta_{ij}\right) L' + E_{kl}\frac{r_k r_l}{r^2}\left(\frac{r_i r_j}{r^2} - \tfrac{1}{3}\delta_{ij}\right) M' \right\}, \tag{2.4}$$

where the quantities $K'$, $L'$ and $M'$ are functions only of the scalar variables $r/a$ and $b/a$. There is a similar expression for the force dipole strength $S''_{ij}$ of the second sphere of radius $b$, and $K'(b/a) \equiv K''(a/b)$, etc.

We shall make some use of the solution of the Stokes equations for a single rigid force-free sphere of radius $a$ with centre at $\mathbf{x}_0$ in fluid whose velocity in the absence of the sphere is $\mathbf{U}$ with uniform gradient at infinity characterized by $E_{ij}$ and $\mathbf{\Omega}$. This isolated sphere rotates with angular velocity $\mathbf{\Omega}$ and its centre has velocity $\mathbf{U}(\mathbf{x}_0)$, and the expressions for the velocity $\mathbf{u}$ and pressure $p$ at position $\mathbf{x}$ in the surrounding fluid are known to be

$$\left.\begin{aligned} u_i(\mathbf{x}) &= U_i(\mathbf{x}_0) + E_{ij}X_j + \epsilon_{ijk}\Omega_j X_k + \hat{u}_i(\mathbf{X}, a) \\ p(\mathbf{x}) &= p_0 - \mu E_{ij}\frac{X_i X_j}{X^2}\frac{5a^3}{X^3}, \end{aligned}\right\} \tag{2.5}$$

where $\mathbf{X} = \mathbf{x} - \mathbf{x}_0$, $X = |\mathbf{X}| \geqslant a$, and

$$\hat{u}_i(\mathbf{X}, a) = E_{jk}X_j\left\{-\delta_{ik}\frac{a^5}{X^5} + \frac{X_i X_k}{X^2}\left(-\frac{5a^3}{2X^3} + \frac{5a^5}{2X^5}\right)\right\} \tag{2.6}$$

is the disturbance velocity due to the presence of the sphere. We note also the following derived expressions associated with this case of flow due to a single rigid sphere in infinite fluid which is being strained and rotated:

$$\begin{aligned} e_{ij}(\mathbf{x}) &= \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \\ &= E_{ij}\left(1 - \frac{a^5}{X^5}\right) + E_{kl}\left(\frac{X_i X_k \delta_{jl} + X_j X_k \delta_{il}}{X^2} - \frac{X_k X_l}{X^2}\tfrac{2}{3}\delta_{ij}\right)\left(-\frac{5a^3}{2X^3} + \frac{5a^5}{X^5}\right) \\ &\quad + E_{kl}\frac{X_k X_l}{X^2}\left(\frac{X_i X_j}{X^2} - \tfrac{1}{3}\delta_{ij}\right)\left(\frac{25a^3}{2X^3} - \frac{35a^5}{2X^5}\right), \end{aligned} \tag{2.7}$$

$$(\tfrac{1}{2}\nabla \times \mathbf{u})_i = \Omega_i + \epsilon_{ijk}E_{kl}\frac{X_j X_l}{X^2}\left(\frac{5a^3}{2X^3}\right), \tag{2.8}$$

$$\nabla^2 u_i = E_{ij}X_j\left(-\frac{10a^3}{X^5}\right) + E_{jk}X_i X_j X_k\left(\frac{25a^3}{X^7}\right), \tag{2.9}$$

for $X > a$.

The local stress can be obtained from (2.5) and (2.7) and then we find from (2.1) that the force dipole strength for a single sphere of radius $a$ in infinite fluid is

$$S_{ij} = \tfrac{20}{3}\pi a^3 \mu E_{ij}. \tag{2.10}$$

It follows from (2.4) that

$$K' \to 0, \quad L' \to 0, \quad M' \to 0 \quad \text{as} \quad r/(a+b) \to \infty, \tag{2.11}$$

and likewise for $K''$, $L''$, $M''$.

The velocity disturbance (2.6) due to a single sphere is a limiting form of that due to the presence of spheres in two senses, one as $r/(a+b) \to \infty$, as already remarked, and the other as $b/a \to 0$. And if one sphere is very much smaller than the other, the smaller sphere moves with the local translational and angular velocity of the fluid in the flow field due to the other sphere alone in infinite fluid. Thus we may compare (2.2) and (2.6) to find that

$$A = \frac{5}{2}\frac{a^3}{r^3} - \frac{3}{2}\frac{a^5}{r^5}, \quad B = \frac{a^5}{r^5}, \tag{2.12}$$

when $b/a = 0$. Also, the angular velocity of the large sphere will be just that of the ambient flow, whilst the small sphere will take up the local angular velocity of the fluid in the flow due to a single sphere, whence a comparison of (2.3) and (2.8) shows that

$$C' = 0, \quad C'' = \frac{5a^3}{2r^3}, \tag{2.13}$$

when $b/a = 0$. Similarly, comparison of (2.4) with (2.10) and (2.7) shows that

$$\left.\begin{aligned} &K' = 0, \quad L' = 0, \quad M' = 0, \\ &K'' = -\frac{a^5}{r^5}, \quad L'' = -\frac{5a^3}{2r^3} + \frac{5a^5}{r^5}, \quad M'' = \frac{25a^3}{2r^3} - \frac{35a^5}{2r^5}, \end{aligned}\right\} \tag{2.14}$$

when $b/a = 0$.

Finally, it should be noted again that the expressions (2.2), (2.3) and (2.4) hold for any values of $E_{ij}$ and $\boldsymbol{\Omega}$ (which may depend on time), and that the dimensionless scalar functions appearing in these expressions depend on $r/a$ and $b/a$ only and are independent of the bulk flow. As a consequence, calculations made for particular ambient flow fields, such as pure straining motion or simple shearing, can be used as a source of information about the values of the scalar functions $A$, $B$, etc. In the following three sections the forms of all the scalar functions will be investigated.

In paper II we shall be concerned with spherical particles composed of fluid of viscosity $\mu'$, with a surface tension at the interface which may be either infinite or zero (the particles being spherical only instantaneously in this latter case). It is evident from the derivation that the expressions (2.2), (2.3) and (2.4) are equally valid for two such fluid spheres provided the scalar functions $A$, $B$, etc. are then allowed to depend on $\mu'/\mu$ as well as on $r/a$ and $b/a$.

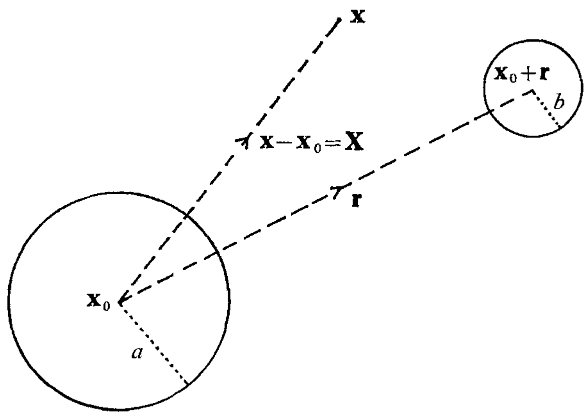


FIGURE 1. Notation used in the investigation of far-field asymptotic forms.

## 3. Far-field analytic forms for the functions

When the distance $r$ between the centres of the two spheres is large compared with either of the two radii $a$ and $b$, the various functions representing the kinematic and dynamic aspects of the hydrodynamic interaction take approximate forms which may be derived by simple methods. These asymptotic expressions are needed for an assessment of the convergence of integrals over an infinite domain which arise in the rheological problem considered in paper II. They prove to be accurate over a surprisingly large part of $\mathbf{r}$-space, and are also useful as a framework for computation.

We begin by considering the functions $A$ and $B$, both functions of $r/a$ and $b/a$ only, which occur in the expression (2.2) for the velocity $\mathbf{V}$ of the centre of a rigid sphere of radius $b$, with instantaneous position $\mathbf{x}_0+\mathbf{r}$, relative to that of a sphere of radius $a$ with instantaneous position $\mathbf{x}_0$. The fluid velocity has a linear variation at infinity characterized by $E_{ij}$ and $\mathbf{\Omega}$ as before. The position vector of an arbitrary point in the fluid is $\mathbf{x}$, and $\mathbf{X} = \mathbf{x}-\mathbf{x}_0$. (The notation is shown also in figure 1.)

The leading approximation to the relative velocity $\mathbf{V}$ when $r/(a+b) \gg 1$ may be obtained from a consideration of the velocity that would exist in the fluid in the absence of one of the spheres. It is known that when a force-free rigid sphere of radius $b$ is placed, with its centre at the point $\mathbf{x}_0+\mathbf{r}$, in *unbounded* fluid in which the velocity is $\mathbf{u}(\mathbf{x})$ (the consequences of the fact that in reality there is a spherical boundary present at $|\mathbf{x}-\mathbf{x}_0| = a$ will be examined in a moment), then under the conditions of the Stokes flow the velocity taken up by the centre of that sphere is exactly

$$\frac{1}{4\pi b^2}\int_{|\mathbf{x}-\mathbf{x}_0-\mathbf{r}|=b} u_i(\mathbf{x})\,dA(\mathbf{x}),\ = \{u_i(\mathbf{x})+\tfrac{1}{6}b^2\nabla^2 u_i(\mathbf{x})\}_{\mathbf{x}=\mathbf{x}_0+\mathbf{r}}. \qquad (3.1)\dagger$$

† This result is sometimes called Faxen's first law; a simple proof is given in the appendix to a recent paper (Batchelor 1972).

We now interpret $\mathbf{u}(\mathbf{x})$ in this expression as the velocity distribution due to the presence of a single sphere of radius $a$ with centre at $\mathbf{x}_0$, whence on substitution from (2.5) the sphere velocity (3.1) becomes

$$U_i(\mathbf{x}_0)+E_{ij}r_j+\epsilon_{ijk}\Omega_j r_k+\{\hat{u}_i(\mathbf{X},a)+\tfrac{1}{6}b^2\nabla^2\hat{u}_i(\mathbf{X},a)\}_{\mathbf{X}=\mathbf{r}}. \tag{3.2}$$

But the introduction of the sphere of radius $b$ with centre at $\mathbf{x}_0+\mathbf{r}$ will itself cause a change in the velocity of the sphere of radius $a$ at $\mathbf{x}_0$ which, under the same restrictions, becomes

$$U_i(\mathbf{x}_0)+\{\hat{u}_i(\mathbf{X},b)+\tfrac{1}{6}a^2\nabla^2\hat{u}_i(\mathbf{X},b)\}_{\mathbf{X}=-\mathbf{r}}. \tag{3.3}$$

The leading approximation to $V_i$ is then the difference between (3.2) and (3.3), and on using (2.6) and (2.9) we find

$$\begin{aligned}V_i(\mathbf{r}) \approx \epsilon_{ijk}\Omega_j r_k+E_{ij}r_j\left\{1-\frac{3(a^5+b^5)+5a^2b^2(a+b)}{3r^5}\right\}\\ +E_{jk}r_ir_jr_k\left\{-\frac{5(a^3+b^3)}{2r^5}+\frac{15(a^5+b^5)+25a^2b^2(a+b)}{6r^7}\right\}.\end{aligned} \tag{3.4}$$

The expression (3.4) has zero divergence with respect to $\mathbf{r}$, and for subsequent application in determining the bulk stress in a suspension it is necessary to take the calculation of $\mathbf{V}$ to one higher order term.

The above expression for $\mathbf{V}$ has been determined in effect by choosing the distribution of force over the surface of each sphere to satisfy the no-slip condition at that surface, taking account of the background linear velocity field and the disturbance velocity due to the other sphere; and the error in it is due wholly to this disturbance velocity being estimated as if that other sphere were alone in infinite fluid. Now the effect of the presence of sphere $a$ with centre at $\mathbf{x}_0$ is to induce, in a region with linear dimensions $b$ centred on $\mathbf{x}_0+\mathbf{r}$, a disturbance velocity which is approximately uniform with magnitude of order $|E_{ij}|\,a^3/r^2$, and a disturbance rate of strain, of order $|E_{ij}|\,a^3/r^3$, given by (2.7). The effect of this induced velocity on sphere $b$ with centre at $\mathbf{x}_0+\mathbf{r}$ is to change its translational velocity by the same amount, since the sphere moves freely; and the effect of this induced rate of strain is to change the net force dipole strength of the sphere $b$ in the manner represented by (2.10). This additional force dipole strength then in turn induces a disturbance velocity at the position of sphere $a$ which is obtainable, to leading order, from the one term of order $(a/X)^2$ in (2.6) with $a$ replaced by $b$, $\mathbf{X}$ replaced by $\mathbf{r}$, and $E_{jk}$ replaced by the terms of order $(a/r)^3$ in the expression (2.7) for the rate of strain induced at position $\mathbf{x}_0+\mathbf{r}$ by the sphere $a$. Bearing in mind again the reciprocal action of one sphere on another, we see that the right-hand side of (3.4) should be corrected by the addition of

$$-\frac{5b^3}{2r^5}r_ir_jr_k\left\{\left(\frac{2E_{jl}r_kr_l+E_{lm}r_lr_m\,\delta_{jk}}{r^2}\right)\left(-\frac{5a^3}{2r^3}\right)+E_{lm}\frac{r_lr_mr_jr_k}{r^4}\left(\frac{25a^3}{2r^3}\right)\right\}$$

$$+\text{similar expression with } a \text{ and } b \text{ exchanged,}$$

$$=-25E_{jk}r_ir_jr_k\frac{a^3b^3}{r^8}. \tag{3.5}$$

Comparison of the general expression (2.2) for $V_i(\mathbf{r})$ with the approximation represented by (3.4) and (3.5) together then shows that for $r \gg a+b$

$$\left.\begin{aligned} A &= \frac{\frac{5}{2}(a^3+b^3)}{r^3} - \frac{\frac{3}{2}(a^5+b^5)+\frac{5}{2}a^2b^2(a+b)}{r^5} + \frac{25a^3b^3}{r^6} + o\left(\frac{(a+b)^6}{r^6}\right), \\ B &= \frac{a^5+b^5+\frac{5}{3}a^2b^2(a+b)}{r^5} + o\left(\frac{(a+b)^6}{r^6}\right). \end{aligned}\right\} \tag{3.6}$$

Asymptotic forms for the relative translational velocity of the centres of two spheres of different sizes in an ambient simple shearing motion have been developed by Wakiya *et al.* (1967), and it may be shown that their results are in accordance with (3.6).

Next we seek asymptotic forms for the scalar functions $K'$, $L'$, $M'$ and $K''$, $L''$, $M''$ occurring in the general expression (2.4) for the force dipole strength of one of the spheres. This can be done by arguments which are analogous to those that led to the Faxen expression (3.1) and to (3.6). The same arguments also yield asymptotic expressions for the angular velocities of the two spheres and of the functions $C'$ and $C''$ occurring in (2.3).

We shall first obtain an expression for the force dipole strength of a rigid sphere of radius $a$ which is placed, with centre at $\mathbf{x}_0$, in fluid in which the velocity before the introduction of that sphere is given by $\mathbf{u}(\mathbf{x})$. If there are no other boundaries to the fluid, the surface density of force ($\mathbf{f}$) which must be applied to the fluid at the surface of the sphere ($A_0$) in order to ensure satisfaction of the no-slip condition there is given exactly by the integral equation:

$$u_i(\mathbf{x}) + \frac{1}{8\pi\mu}\int_{A_0} I_{ij}(\mathbf{x}-\mathbf{x}')f_j(\mathbf{x}')\,dA(\mathbf{x}') = \frac{1}{4\pi a^2}\int_{A_0} u_i(\mathbf{x}')\,dA(\mathbf{x}') + \{\mathbf{\Gamma}\times(\mathbf{x}-\mathbf{x}_0)\}_i \tag{3.7}$$

for all $\mathbf{x}$ on $A_0$, where $\mathbf{\Gamma}$ is the angular velocity taken up by the sphere and

$$\frac{1}{8\pi\mu}I_{ij}(\mathbf{y}) = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{|\mathbf{y}|} + \frac{y_iy_j}{|\mathbf{y}|^3}\right)$$

is the tensor coefficient in the linear relation between the velocity induced in the fluid and a point force applied to the fluid at relative position $\mathbf{y}$. The desired result is obtained by multiplying both sides of (3.7) by $(\mathbf{x}-\mathbf{x}_0)_k$ and integrating over $A_0$. It may be found, by straightforward working, that

$$\int_{A_0}(\mathbf{x}-\mathbf{x}_0)_k\,u_i(\mathbf{x})\,dA(\mathbf{x}) = \frac{4\pi}{3}a^4\left(\frac{\partial u_i}{\partial x_k} + \tfrac{1}{10}a^2\nabla^2\frac{\partial u_i}{\partial x_k}\right)_{\mathbf{x}=\mathbf{x}_0}$$

and

$$\int_{A_0}(\mathbf{x}-\mathbf{x}_0)_k I_{ij}(\mathbf{x}-\mathbf{x}')\,dA(\mathbf{x}) = \frac{8\pi a}{15}\{4(\mathbf{x}'-\mathbf{x}_0)_k\delta_{ij} - (\mathbf{x}'-\mathbf{x}_0)_i\delta_{jk} - (\mathbf{x}'-\mathbf{x}_0)_j\delta_{ki}\};$$

and then the symmetric and antisymmetric parts of the integrated form of (3.7) become, for a couple-free sphere whose force dipole strength is defined by (2.1),

$$S_{ij} = \tfrac{20}{3}\pi a^3\mu(e_{ij} + \tfrac{1}{10}a^2\nabla^2 e_{ij})_{\mathbf{x}=\mathbf{x}_0}, \tag{3.8}$$

$$\mathbf{\Gamma} = \tfrac{1}{2}(\nabla\times\mathbf{u})_{\mathbf{x}=\mathbf{x}_0}, \tag{3.9}$$

where $e_{ij}(\mathbf{x})$ is the rate of strain corresponding to the velocity distribution $\mathbf{u}(\mathbf{x})$. (The relation (3.9) is the second Faxen law, for a couple-free sphere; but (3.8) appears to be new.)

The rate of strain $e_{ij}$ in (3.8) and vorticity $\nabla\times\mathbf{u}$ in (3.9) may now be interpreted as that due to the presence of a single sphere of radius $b$ with centre at $\mathbf{x}_0+\mathbf{r}$ in infinite fluid with uniform velocity gradient at infinity, in which case the values of $e_{ij}$, $\nabla^2 e_{ij}$ and $\nabla\times\mathbf{u}$ at position $\mathbf{x}_0$ are obtained from (2.7) and (2.8) by replacing $a$ and $\mathbf{X}$ by $b$ and $-\mathbf{r}$ respectively (after differentiation, in the case of $\nabla^2 e_{ij}$). In this way we find for the force dipole strength and angular velocity of sphere $a$ in the presence of sphere $b$

$$S'_{ij}\approx\tfrac{20}{3}\pi a^3\mu\left\{E_{ij}\left(1-\frac{b^3(a^2+b^2)}{r^5}\right)+E_{kl}\left(\frac{r_jr_k}{r^2}\delta_{il}+\frac{r_kr_i}{r^2}\delta_{jl}-\frac{r_kr_l}{r^2}\tfrac{2}{3}\delta_{ij}\right)\right.$$
$$\left.\times\left(-\frac{5b^3}{2r^3}+\frac{5b^3(a^2+b^2)}{r^5}\right)+E_{kl}\frac{r_kr_l}{r^2}\left(\frac{r_ir_j}{r^2}-\tfrac{1}{3}\delta_{ij}\right)\left(\frac{25b^3}{2r^3}-\frac{35b^3(a^2+b^2)}{2r^5}\right)\right\} \tag{3.10}$$

and
$$\Gamma'_i\approx\Omega_i+\epsilon_{ijk}E_{kl}r_jr_l\frac{5b^3}{2r^5}. \tag{3.11}$$

The errors in (3.10) and (3.11) arise wholly from the fact that the effect of the presence of the sphere $b$ has been estimated as if that sphere were alone in infinite fluid. An improved approximation to $S'_{ij}$ may be obtained by noting that, as a consequence of the presence of the sphere $a$ with centre at $\mathbf{x}_0$, the approximately uniform rate of strain in the fluid in which the sphere $b$ is immersed exceeds $E_{ij}$ by an amount shown by (2.7) to be

$$E_{kl}\frac{r_jr_k\delta_{il}+r_ir_k\delta_{jl}+r_kr_l\delta_{ij}}{r^2}\left(-\frac{5a^3}{2r^3}\right)+E_{kl}\frac{r_ir_jr_kr_l}{r^4}\frac{25a^3}{2r^3}+O\left(\frac{a^5}{r^5}\right).$$

The disturbance due to the sphere $b$ is correspondingly greater, and the rate of strain that the sphere $b$ induces at the location of the centre of sphere $a$ and which is to be substituted in (3.8) is greater than that previously estimated by the amount

$$E_{mn}\left\{\frac{r_jr_k\delta_{il}+r_ir_k\delta_{jl}+r_kr_l\delta_{ij}}{r^2}\left(-\frac{5b^3}{2r^3}\right)+\frac{r_ir_jr_kr_l}{r^4}\frac{25b^3}{2r^3}\right\}$$
$$\times\left\{\frac{r_kr_m\delta_{ln}+r_lr_m\delta_{kn}+r_mr_n\delta_{kl}}{r^2}\left(-\frac{5a^3}{2r^3}\right)+\frac{r_kr_lr_mr_n}{r^4}\frac{25a^3}{2r^3}\right\},$$

correct to the order $(a+b)^6/r^6$. The correction to the expression (3.10) for $S'_{ij}$ is then $\tfrac{20}{3}\pi a^3\mu$ times this, and is found, after multiplying out, to be

$$\tfrac{20}{3}\pi a^3\mu E_{mn}\left(\frac{r_ir_m}{r^2}\delta_{jn}+\frac{r_jr_m}{r^2}\delta_{in}-\frac{2r_mr_n}{r^2}\delta_{ij}+\frac{4r_ir_jr_mr_n}{r^4}\right)\frac{25a^3b^3}{4r^6}. \tag{3.12}$$

We may now obtain an expression for $S'_{ij}$ which is correct to the order of $(a+b)^6/r^6$ by adding (3.12) to the right-hand side of (3.10). The asymptotic forms for the functions $K'$, $L'$ and $M'$ defined by (2.4) are then seen to be

$$K'=-\frac{b^3(a^2+b^2)}{r^5}+o\left(\frac{(a+b)^6}{r^6}\right), \tag{3.13}$$

$$L' = -\frac{5b^3}{2r^3} + \frac{5b^3(a^2+b^2)}{r^5} + \frac{25a^3b^3}{4r^6} + o\left(\frac{(a+b)^6}{r^6}\right), \tag{3.14}$$

$$M' = \frac{25b^3}{2r^3} - \frac{35b^3(a^2+b^2)}{2r^5} + \frac{25a^3b^3}{r^6} + o\left(\frac{(a+b)^6}{r^6}\right). \tag{3.15}$$

The corresponding asymptotic forms for $K''$, $L''$, $M''$ are obtained by interchanging $a$ and $b$ in (3.13), (3.14) and (3.15).

In exactly the same way we can improve the approximation (3.11) to the angular velocity of one of the spheres, and then comparison with (2.3) shows that the asymptotic form of the function $C'$ when $(a+b)/r \ll 1$ is

$$C' = \frac{5b^3}{2r^3} - \frac{25a^3b^3}{4r^6} + o\left(\frac{(a+b)^6}{r^6}\right), \tag{3.16}$$

with that for $C''$ being obtained by interchanging $a$ and $b$ in (3.16). Again there is agreement with the asymptotic expressions for the angular velocities of two spheres of different sizes in an ambient simple shearing motion developed by Wakiya *et al.* (1967).

Further terms in the above expansions of two-sphere functions in powers of $1/r$ can of course be obtained by iterative methods. But, to judge by experience with some other two-sphere quantities, such as the common translational velocity of two equal spheres moving through fluid under the action of equal external forces, such expansions are unlikely to be useful when the two spheres are close together. It is necessary to employ other ways of calculating the functions in the range in which $r/(a+b)$ is of order unity, and the above outer asymptotic forms provide only an analytical supplement to that more extensive and inevitably mainly numerical computation.

## 4. Data obtained from full solutions for the velocity distribution due to two spheres in particular ambient flow fields

In this section we use several exact calculations which have been made by previous authors for interactions between two spheres in particular ambient flow fields, as a source of information about the forms of the scalar functions $A$, $B$, $C'$ and $C''$. Much of this information refers to the case of spheres of equal size. Unfortunately the calculations to be described in this section tell us nothing about the functions $K'$, $L'$, $M'$, $K''$, $L''$, $M''$, and our knowledge of these functions is at present restricted to their far-field asymptotic forms given in § 3 and to some information which will be derived in § 5 about their behaviour when the spheres are touching (and that only for $b/a = 1$).

### 4.1. *Two spheres in a steady simple shearing motion*

In a recent extensive investigation of the kinematics of a two-sphere encounter, Lin *et al.* (1970) give an exact series expression for the velocity distribution in terms of bispherical co-ordinate solutions of Laplace's equation for the case of two force-free rigid spheres of arbitrary radius ratio in a steady simple shearing

| $r/a$ | $A(r)$ | $B(r)$ | $C'(r)$, $C''(r)$ |
|---|---|---|---|
| 20·1353 | 0·0006 | 0·0000 | 0·0003 |
| 11·1139 | 0·0036 | 0·0000 | 0·0018 |
| 6·2149 | 0·0204 | 0·0006 | 0·0103 |
| 4·7048 | 0·0468 | 0·0023 | 0·0234 |
| 3·6213 | 0·1033 | 0·0086 | 0·0501 |
| 3·3370 | 0·1331 | 0·0130 | 0·0633 |
| 3·0862 | 0·1704 | 0·0193 | 0·0791 |
| 2·8662 | 0·2167 | 0·0281 | 0·0976 |
| 2·6749 | 0·2735 | 0·0399 | 0·1190 |
| 2·5103 | 0·3424 | 0·0553 | 0·1433 |
| 2·3709 | 0·4248 | 0·0748 | 0·1709 |
| 2·2553 | 0·5214 | 0·0988 | 0·2019 |
| 2·1621 | 0·6313 | 0·1275 | 0·2369 |
| 2·0907 | 0·7505 | 0·1608 | 0·2767 |
| 2·0401 | 0·8679 | 0·1996 | 0·3230 |
| 2·0100 | 0·9619 | 0·2461 | 0·3806 |
| 2·0025 | 0·9900 | 0·2762 | 0·4195 |
| 2·0006 | 0·9975 | 0·2968 | 0·4469 |
| 2·0001 | 0·9996 | (0·4426) | (0·3265) |
| 2·00006 | 0·9998 | (0·7266) | (0·0407) |

TABLE 1. Values of the functions $A$, $B$, $C'$ and $C''$ describing the velocities of two equal spheres derived from the numerical data of Lin *et al.* (1970). The data in parentheses are inconsistent with the asymptotic forms found in § 5.

motion. The relative velocity of the two sphere centres and the angular velocities of the two spheres are not obtained explicitly, but they can be computed and the authors give numerical results for the case of equal sphere sizes. The functions $A$, $B$, $C'$ and $C''$ defined by (2.2) and (2.3) are independent of the bulk flow, and the numerical data presented by Lin *et al.* can be interpreted to yield the values of these functions. A comparison of the expressions for $\mathbf{V}$, $\mathbf{\Gamma}'$ and $\mathbf{\Gamma}''$ given by (2.2) and (2.3) for the particular case of a simple shearing motion with the expressions given by Lin *et al.*, which are described in terms of several functions of $r$ denoted by script letters, shows the following relations between the two sets of functions (for $b/a = 1$):

$$A = 1-\frac{\mathscr{C}}{r}, \quad B = -\frac{2\mathscr{A}}{r}, \quad C' = C'' = \mathscr{E}-1. \tag{4.1}$$

The comparison also shows that the functions used by Lin *et al.* must satisfy the identities

$$\mathscr{B}-\mathscr{A} = r, \quad \mathscr{E}-\mathscr{D} = 2; \tag{4.2}$$

their tabulated numerical values confirm this.

The values of the functions $A$, $B$, $C'$ and $C''$ derived from the data of Lin *et al.* using (4.1) are shown in table 1 and figure 2. The far-field forms of these functions found in § 3 are also shown in figure 2, and are seen to be accurate for remarkably small values of the separation of the spheres.

The values given in table 1 suggest that the gradients of the functions $B$ and $C'$, $C''$ are very large at $r/a = 2$ (for this case of equal spheres). There is evidently

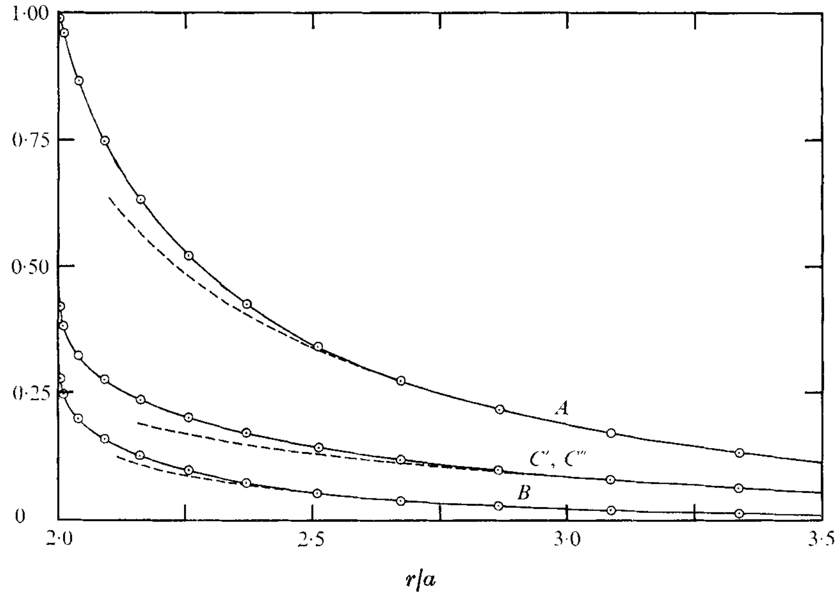


FIGURE 2. Graphs of the functions $A$, $B$, $C'$ and $C''$ describing the velocities of two equal spheres. The encircled points correspond to the numerical data of table 1. The broken lines show the far-field asymptotic forms of these functions found in § 3.

need for an investigation of the analytical forms of these functions near $r/a = 2$. We shall provide this in § 5. The singular behaviour of some of the functions at $r/a = 2$ renders the calculations of Lin *et al.* less accurate when the spheres are very nearly touching, and the derived values of these functions at $r/a = 2{\cdot}00006$ and $2{\cdot}0001$, shown bracketed in table 1, are certainly inconsistent with the asymptotic forms found in § 5.

This full solution for the velocity distribution obtained by Lin *et al.* could also be used for the computation of the functions $K'$, $L'$, $M'$, $K''$, $M''$, $N''$ describing the force dipole strengths of the two spheres. We understand from Prof. Sather that such computations are now being made.

### 4.2. *Two equal spheres in an axisymmetric flow system*

The results of an exact solution for the flow due to two equal rigid spheres in a more restricted type of ambient flow field will be recorded here. (Details of the solution have been given elsewhere by Green 1971.) The sphere centres lie on the axis of a steady axisymmetric pure straining motion which is characterized by

$$E_{11} = E_{22} = -\tfrac{1}{2}G, \quad E_{33} = G, \quad E_{ij} = 0 \text{ if } i \neq j, \tag{4.3}$$

and $\mathbf{\Omega} = 0$. This solution can provide information about only one of the functions $A, B, C', C''$, since, as a consequence of the chosen geometry, the angular velocities of the two spheres are zero and the circumferential component of $\mathbf{V}$ (which involves $B$) is zero. Likewise it can give only one linear relation between the three functions $K'$, $L'$, $M'$ (or $K''$, $L''$, $M''$).

The solution was obtained by the same method employing bispherical coordinates as was used by Lin *et al.* (1970). The Stokes stream function of the fluid motion is written as an infinite series of bispherical harmonics, the coefficients of which are determined by the no-slip condition at the surfaces of the two spheres. It is then possible to derive an exact relation between the total force $(0, 0, \mp F_z)$ on a sphere and an assumed relative velocity $(0, 0, V_z)$ of the two spheres. The value of the solution lies in the fact that the analytical form of this relation between $F_z$ and $V_z$ allows evaluation near the singular limit when the spheres are touching, thereby giving values of $A(r)$ in a range where the data from Lin *et al.* (1970) are not accurate. It is found that when $\xi \ll 1$

$$F_z = -\tfrac{3}{2}\pi a\mu V_z(\xi^{-1} + \tfrac{9}{10}\log\xi^{-1} + 2{\cdot}763) + 12{\cdot}233\pi a^2\mu G + O(\xi^{\frac{1}{2}}), \tag{4.4}$$

where $\xi = (r-2a)/a$, and the corresponding expression for $V_z$ obtained by setting the force on each sphere equal to zero is

$$V_z = 8{\cdot}155 aG\xi\{1 + O(\xi^{\frac{1}{2}})\}. \tag{4.5}$$

The form taken by the general expression (2.2) for the radial component of the relative velocity $\mathbf{V}$ when $\mathbf{r} = (0, 0, r)$ and the components of $E_{ij}$ are as in (4.3) is $rG(1-A)$, whence it follows that

$$A(r) = 1 - 4{\cdot}077\xi + O(\xi^{\frac{3}{2}}) \tag{4.6}$$

when $\xi \ll 1$ and $b/a = 1$.

The additional rate of energy dissipation due to the presence of the two spheres in this axisymmetric flow field can also be evaluated, and is found to be

$$38{\cdot}21\pi a^3\mu G^2\{1 + O(\xi)\} \tag{4.7}$$

when $\xi \ll 1$. The general expression for the additional rate of dissipation is $2E_{ij}S'_{ij}$, where $S'_{ij}$ is given by (2.4), and on substituting the special values of the components of $r_i$ and $E_{ij}$ and equating to (4.7) we find

$$\tfrac{3}{2}K' + 2L' + M' = 1{\cdot}366 + O(\xi) \tag{4.8}$$

when $\xi \ll 1$ and $b/a = 1$.

Far-field asymptotic forms of the functions $A(r)$ and $\frac{3}{2}K' + 2L' + M'$ have also been deduced from this axisymmetric flow field, but these merely confirm (3.6) and the appropriate combinations of (3.13), (3.14) and (3.15).

### 4.3. *A sphere in the presence of a plane wall*

We first make use of an exact solution given by Goldman *et al.* (1967*b*) for the motion of a force-free couple-free rigid sphere in a simple shearing motion in the presence of a stationary plane rigid wall in order to find the forms of the functions $B$ and $C''$ when $b/a \ll 1$. If one of our two spheres if very much smaller than the other, the small sphere is effectively immersed in the flow due to the large sphere alone, that is, in the flow field given by (2.5) and (2.6). Near a point on the surface of the large sphere of radius $a$ where the unit normal is $\mathbf{n}$, the velocity relative to the surface in the absence of the small sphere is

$$u_i \approx 5(X-a)\,E_{jk}n_j(\delta_{ik} - n_i n_k) \tag{4.9}$$

| $\dfrac{r-a}{b}$ | $(1-A)\dfrac{a^2}{b^2}\Big/\dfrac{15(r-a)^2}{2b^2}$ | $(1-B)\dfrac{a}{b}\Big/\dfrac{5(r-a)}{b}$ | $C''$ |
|---|---|---|---|
| 10 | 1·0015 | 0·9996 | 2·4988 |
| 8 | 1·0016 | — | — |
| 6 | 1·0008 | — | — |
| 4 | 0·9932 | — | — |
| 3·7622 | — | 0·9944 | 2·4858 |
| 2·5 | 0·9475 | — | — |
| 2·3524 | — | 0·9777 | 2·4445 |
| 1·5431 | — | 0·9219 | 2·3092 |
| 1·5 | 0·7135 | — | — |
| 1·1276 | — | 0·7669 | 1·9479 |
| 1·1 | 0·2579 | — | — |
| 1·05 | 0·1429 | — | — |
| 1·0453 | — | 0·6538 | 1·6866 |
| 1·01 | 0·03139 | — | — |
| 1·0050 | 0·01591 | 0·4786 | 1·2705 |
| 1·0032 | — | 0·4529 | 1·2075 |
| 1·001 | 0·00322 | — | — |

TABLE 2. Values of the relative velocity functions for the case $b/a \ll 1$, deduced from the data of Goldman, Cox & Brenner (1967*b*) in the case of $B$ and $C''$ and from that of Goren & O'Neill (1971) in the case of $A$.

when $X-a \ll a$, which represents (locally) a simple shearing motion with a velocity gradient of magnitude

$$\kappa = 5\{E_{ij}E_{kl}n_i n_k(\delta_{jl}-n_j n_l)\}^{\frac{1}{2}}. \tag{4.10}$$

A small sphere of radius $b$ with centre at distance $r-a$ ($\ll a$) from the surface in this simple shearing motion moves parallel to the surface with a translational velocity relative to the surface whose magnitude is seen from (2.2) to be of the form

$$W = r(1-B)\{E_{ij}E_{kl}n_i n_k(\delta_{jl}-n_j n_l)\}^{\frac{1}{2}} \approx \tfrac{1}{5}\kappa a(1-B), \tag{4.11}$$

and with a rotational velocity relative to that of the big sphere (which is approximately $\mathbf{\Omega}$, as noted in (2.13)) whose magnitude is seen from (2.3) to be approximately

$$\omega = \tfrac{1}{5}\kappa C''. \tag{4.12}$$

By combining the exact solution for the translation and rotation of a sphere near a wall in fluid at rest at infinity with that for an immobilized sphere near a plane wall in a simple shearing motion, Goldman *et al.* (1967*b*) obtain expressions for the translational velocity $W$ and angular velocity $\omega$ of a force-free couple-free sphere with centre at distance $h$ from a plane wall in a simple shearing motion with velocity gradient $\kappa$. They evaluate $W/b\kappa$ and $\omega/\kappa$ numerically as functions of $h$, and the corresponding values of the functions $B$ and $C''$ obtained from (4.11) and (4.12) by identifying $h$ with $r-a$ are indicated in table 2.

An exact solution which may be used in a similar way to obtain values of the function $A$ for $b/a \ll 1$ is that given by Goren & O'Neill (1971) for the velocity of a rigid sphere driven toward a plane rigid boundary by a stagnation-point flow. We find from (2.5) and (2.6) that, near a point on the surface of the sphere $a$

where the unit normal is $\mathbf{n}$, the component of velocity relative to and normal to the surface in the absence of sphere $b$ is approximately

$$\tfrac{15}{2}a\left(\frac{X-a}{a}\right)^2 E_{ij}n_i n_j$$

when $X-a \ll a$. And from our general expression (2.2) for the relative velocity of the two sphere centres we see that the component of this relative velocity parallel to the line of centres is approximately

$$a(1-A)\,n_i n_j E_{ij}$$

when $b$ and $r-a$ are small compared with $a$. The ratio of these two quantities (with $X$ equated to $r$) is equal to the ratio of the functions $f$ and $\bar{f}_0$, values of which are given in table 3 of the paper by Goren & O'Neill, whence we obtain the values of $A$ given in table 2 above.

We note that the values of the functions $A$, $B$ and $C''$ for large $(r-a)/b$ (but still such that $(r-a)/a \ll 1$) are consistent with the forms found in (2.12) and (2.13) for this case $b/a \ll 1$.

These exact solutions of Goldman *et al.* (1967*b*) and Goren & O'Neill (1971) could also provide some information about the functions $K''$, $L''$ and $M''$, but the necessary computations have not yet been carried out.

## 5. Near-field analytic forms for the functions

Asymptotic forms of the functions representing the hydrodynamic interaction may also be found for distances between the centres of two spheres which are very close to the limiting value $a+b$, although these are less complete than the far-field asymptotic forms derived in § 3. As already noted, some of the functions have gradients which are singular at $r = a+b$, and it has been found necessary in paper II to use these asymptotic forms rather than numerical data for values of $r$ close to $a+b$.

The calculation of the asymptotic forms of the functions $A$, $B$, $C'$ and $C''$ occurring in the expressions (2.2) and (2.3) rests on the familiar idea of representing the actual flow field as a superposition of two auxiliary flow systems. In one of these the two spheres together move as a force-free couple-free rigid dumb-bell in fluid which has the specified uniform rate of strain $E_{ij}$ and angular velocity $\mathbf{\Omega}$ at infinity. In the other the two spheres are forced by external agencies to have their actual additional motions in fluid which is at rest at infinity. In the first of these auxiliary flow systems the fluid exerts a force $\mathbf{F}^{(A)}$ and a couple $\mathbf{G}^{(A)}$ (about the centre) on one sphere, say the sphere $b$, due solely to the ambient flow; and in the other the fluid exerts a force $\mathbf{F}^{(S)}$ and a couple $G^{(S)}$ (about the centre) on sphere $b$ due solely to the motion of the spheres relative to the force-free couple-free dumb-bell. The actual translational and rotational velocities of each of the spheres must then satisfy the pair of equations

$$\mathbf{F}^{(S)}+\mathbf{F}^{(A)} = 0, \quad \mathbf{G}^{(S)}+\mathbf{G}^{(A)} = 0. \tag{5.1}$$

When the two spheres are nearly touching $\mathbf{F}^{(S)}$ and $\mathbf{G}^{(S)}$ are dominated by the contributions from the stresses of large magnitude in the thin layer between the two neighbouring sections of the sphere surfaces due to the forced relative motion of these two surfaces. This relative motion of the two surfaces is described most conveniently in terms of a moving Cartesian co-ordinate system $(x, y, z)$ with the origin permanently at the centre of sphere $a$, all three axes permanently fixed in sphere $a$, and the $z$-axis instantaneously in the direction of the line of centres. The components of the vector $\mathbf{r}$ separating the centres of the two spheres will be written as

$$0, 0, (a+b)(1+\tfrac{1}{2}\xi).$$

The components of the velocity of the centre of sphere $b$ are then

$$V_x - (a+b)(1+\tfrac{1}{2}\xi)\Gamma'_y, \quad V_y + (a+b)(1+\tfrac{1}{2}\xi)\Gamma'_x, \quad V_z,$$

and the angular velocity of this sphere is $\mathbf{\Gamma}'' - \mathbf{\Gamma}'$. It may be shown by the methods of lubrication theory that when $\xi \ll 1$ the components of the hydrodynamic force exerted on sphere $b$ due to the relative motion of the two spheres are approximately

$$F_x^{(S)} = \mu(a+b)[-\alpha\{V_x - (a+b)\Gamma'_y\} + \beta(a+b)(\Gamma''_y - \Gamma'_y)]\log\xi^{-1}, \qquad (5.2')$$

$$F_y^{(S)} = \mu(a+b)[-\alpha\{V_y + (a+b)\Gamma'_x\} - \beta(a+b)(\Gamma''_x - \Gamma'_x)]\log\xi^{-1}, \qquad (5.2'')$$

$$F_z^{(S)} = -\frac{12\pi a^2 b^2 \mu}{(a+b)^3}\frac{V_z}{\xi}, \qquad (5.2''')$$

where $\alpha$ and $\beta$ are dimensionless functions of $b/a$ alone with values of order unity in general. Similarly, the $x$ and $y$ components of the couple on the sphere $b$ about its centre due the relative motion of the two spheres are found to be approximately

$$G_x^{(S)} = \mu(a+b)^2[-\beta\{V_y + (a+b)\Gamma'_x\} - \gamma(a+b)(\Gamma''_x - \Gamma'_x)]\log\xi^{-1}, \qquad (5.3')$$

$$G_y^{(S)} = \mu(a+b)^2[\beta\{V_x - (a+b)\Gamma'_y\} - \gamma(a+b)(\Gamma''_y - \Gamma'_y)]\log\xi^{-1}, \qquad (5.3'')$$

where $\gamma$ is a dimensionless function of $b/a$ alone with value of order unity in general.† The neglected terms in (5.2) and (5.3) are necessarily linear in the relative translational and angular velocities of the adjoining surfaces of the two spheres, and so vanish with $\xi$ in the case of spheres which are force-free and couple-free.

Now the force and couple exerted by hydrodynamic stresses on one of the two spheres moving as a force-free couple-free rigid dumb-bell in fluid with rate of

† The identity of the coefficients represented by $\beta$ in (5.2′) and (5.2″) on the one hand, and in (5.3′) and (5.3″) on the other, follows from the same argument as is used (see Happel & Brenner 1965, chap. 5) to establish identity of the corresponding coefficients for a body in translational and rotational motion through fluid which is otherwise unbounded. The values of $\alpha$, $\beta$ and $\gamma$ have been found explicitly by O'Neill & Majumdar (1970) to be as follows:

$$\alpha = \tfrac{8}{5}\pi\lambda\frac{2+\lambda+2\lambda^2}{(1+\lambda)^4}, \quad \beta = \tfrac{4}{5}\pi\lambda^2\frac{1+4\lambda}{(1+\lambda)^4}, \quad \gamma = \tfrac{16}{5}\pi\lambda^3\frac{1}{(1+\lambda)^4},$$

where $\lambda = b/a$.

strain $E_{ij}$ and angular velocity $\mathbf{\Omega}$ at infinity can depend only on $a$, $b$, $\mu$, $E_{ij}$ and $\mathbf{r}$, and so an argument like that used to obtain (2.2), (2.3) and (2.4) shows that

$$F_i^{(A)} = \mu(a+b)\, r_k E_{jk} \left\{ \frac{r_i r_j}{r^2} P + \left( \delta_{ij} - \frac{r_i r_j}{r^2} \right) Q \right\}, \tag{5.4}$$

$$G_i^{(A)} = \mu(a+b)^2 \epsilon_{ijk} E_{kl} \frac{r_j r_l}{r^2} R, \tag{5.5}$$

where $P$, $Q$ and $R$ are functions only of $r/a$ and $b/a$. It is important to note that there are no singularities in the stress in this auxiliary flow system in which the two spheres move as a rigidly connected pair, so that the functions $P$, $Q$ and $R$ have values of order unity however close $r$ may be to its limiting value $a+b$. The component of $\mathbf{F}^{(A)}$ along the line of centres of the two spheres is then

$$\begin{aligned} F_z^{(A)} = \frac{\mathbf{r}\,.\,\mathbf{F}^{(A)}}{r} &= \mu(a+b)\, r E_{zz} P, \\ &\approx \mu(a+b)^2 E_{zz} P_0 \end{aligned} \tag{5.6}$$

when $\xi \ll 1$, where $P_0$ is the value of $P$ at $r = a+b$ and $E_{zz} = E_{jk} r_j r_k / r^2$. Expressions for the other components of $\mathbf{F}^{(A)}$ and $\mathbf{G}^{(A)}$ in the directions of the $x$, $y$ and $z$ axes may be found in a similar way.

We may now use the balance equations (5.1) to determine the relative motion of the surfaces of the two spheres and thereby to find the limiting forms of $A$, $B$, $C'$ and $C''$ as $\xi \to 0$. From (2.2) we see that the general form of the component of $\mathbf{V}$ along the line of centres is

$$\begin{aligned} V_z = \frac{\mathbf{r}\,.\,\mathbf{V}}{r} &= \frac{r_i r_j}{r} E_{ij}(1-A), \\ &\approx (a+b)\, E_{zz}(1-A) \end{aligned}$$

when $\xi \ll 1$. It follows then from (5.2‴) and (5.6) that the required asymptotic form of $A$ is

$$A = 1 - \frac{(a+b)^4 P_0}{12\pi a^2 b^2} \xi + o(\xi). \tag{5.7}$$

Similarly, we find from (5.2′) and (5.4) that

$$\alpha(1-B-C') - \beta(C''-C') = \frac{Q_0}{\log \xi^{-1}}, \tag{5.8}$$

where $Q_0$ is the value of $Q$ at $r = a+b$. An identical relation results from the balance between $F_y^{(S)}$ and $F_y^{(A)}$. The required balance between the two contributions to the couple on one sphere, $G_x^{(S)}$ and $G_x^{(A)}$, yields the further relation

$$\beta(1-B-C') + \gamma(C''-C') = -\frac{R_0}{\log \xi^{-1}}, \tag{5.9}$$

and the same relation comes from the $y$-components. From (5.8) and (5.9) we find that when $\xi \ll 1$

$$C'' = C_0 + O\left(\frac{1}{\log \xi^{-1}}\right), \quad C' = C_0 + O\left(\frac{1}{\log \xi^{-1}}\right), \tag{5.10}$$

where $C_0$ is the common value of $C''$ and $C'$ at $r = a+b$, and

$$B = 1 - C_0 + O\left(\frac{1}{\log \xi^{-1}}\right). \tag{5.11}$$

The asymptotic forms (5.7), (5.10) and (5.11) contain the two parameters $P_0$ and $C_0$ which depend only on the radius ratio $b/a$.

We note incidentally that the gradients of $B$, $C'$ and $C''$ are infinite at $\xi = 0$, as surmised in § 4.1 from the numerical data given in table 1 for the case $b/a = 1$.

There seems to be no reason to expect the functions $K'$, $L'$, $M'$, $K''$, $L''$, $M''$ to have singular properties at $r = a+b$, and presumably they all approach finite limiting values smoothly as $r \to a+b$. (The relation (4.8) derived from the exact solution for two equal spheres in an axisymmetric pure straining motion shows that one linear combination of $K'$, $L'$, $M'$ certainly has a finite gradient at $\xi = 0$.) The additional rate of dissipation of energy due to the presence of the two spheres in the linear ambient flow field is

$$E_{ij}(S'_{ij}+S''_{ij}) = \tfrac{20}{3}\pi\mu\bigg[E_{ij}E_{ij}\{a^3(1+K')+b^3(1+K'')\} + E_{ij}E_{jk}\frac{r_i r_k}{r^2}(2a^3L'+2b^3L'') + E_{ij}E_{kl}\frac{r_i r_j r_k r_l}{r^4}(a^3M'+b^3M'')\bigg], \tag{5.12}$$

and so a calculation of this additional rate of dissipation for three different orientations of a pair of touching spheres (which would move as a rigid body) in any linear ambient flow field would provide the values of $a^3K'+b^3K''$, $a^3L'+b^3L''$ and $a^3M'+b^3M''$ at $r = a+b$. Unless $b/a = 1$ or $b/a \ll 1$, three more numerical relations must be obtained in some way to allow the values of $K'$ and $K''$, etc. to be determined separately, although for the purpose of calculating the bulk stress in a suspension of spherical particles of non-uniform size only the combinations like $a^3K'+b^3K''$ are required. The one calculation of the above type which has been carried out for a particular value of $b/a$ is mentioned below.

### *Calculations for the case $b/a = 1$*

The values of all the parameters appearing in the above asymptotic forms can be given for the case of two spheres of the same size.

In § 4.2 we saw from an exact solution for the movement of two force-free equal spheres along the axis of an axisymmetric pure straining motion that $F_z^{(S)}$ and $F_z^{(A)}$ do indeed have the forms given in (5.2‴) and (5.6) when the gap between the sphere surfaces is small, and that

$$P_0 = 3{\cdot}058\pi; \tag{5.13}$$

this exact solution also shows that the error term in (5.7) is of order $\xi^{\frac{3}{2}}$. The expression for $A$ in the case $b/a = 1$ is therefore

$$A(r) = 1 - 4{\cdot}077\xi + O(\xi^{\frac{3}{2}})$$

when $\xi = (r-2a)/a \ll 1$, as already noted in (4.6).

Wakiya (1971) has recently determined the motion of a force-free couple-free doublet of two equal spheres in contact (and moving as a single rigid body)

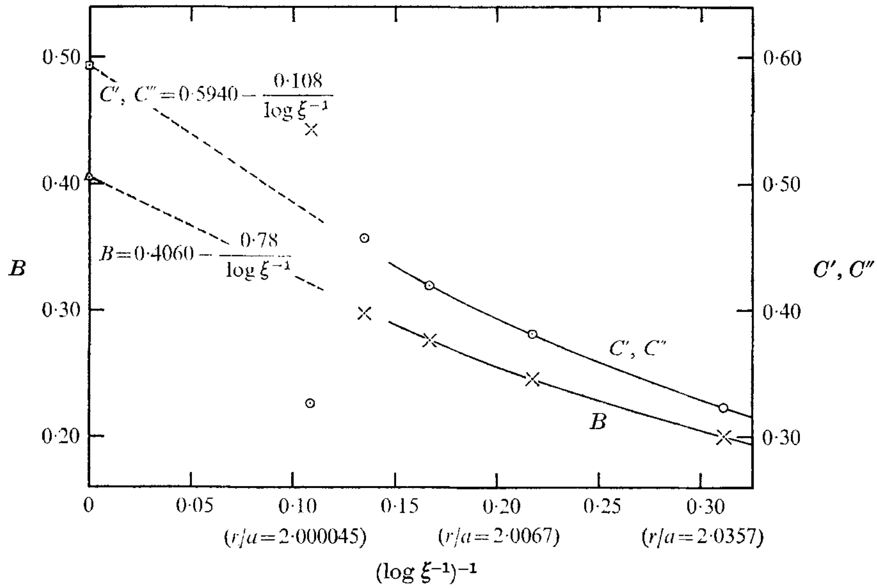


FIGURE 3. Graphs of the functions $B$, $C'$ and $C''$ for the case of two equal spheres for small values of $\xi(=(r-2a)/a)$. The marked points at $\xi = 0$ ($\triangle$ for $B$, $\square$ for $C'$, $C''$) are derived from the near-field theory of §5 and are accurate. The other marked points ($\times$ for $B$, $\odot$ for $C'$, $C''$) correspond to the computed values shown in table 1, the accuracy of which decreases as $\xi \to 0$. The broken lines are near-field asymptotic forms, with the coefficients of $(\log \xi^{-1})^{-1}$ chosen on the basis of consistency with the more reliable of the values from table 1.

immersed in a simple shearing flow, and from his expression for the common angular velocity of the two spheres we see that

$$C_0 = 0{\cdot}5940. \tag{5.14}$$

The asymptotic forms for $B$, $C'$ and $C''$ are consequently

$$B(r) = 0{\cdot}4060 + O\left(\frac{1}{\log \xi^{-1}}\right), \tag{5.15}$$

$$C'(r) = C''(r) = 0{\cdot}5940 + O\left(\frac{1}{\log \xi^{-1}}\right) \tag{5.16}$$

in the case $b/a = 1$.

For the purposes of the rheological calculation in paper II, it is desirable to have an improved approximation to the function $B(r)$ near $\xi = 0$. The explicit determination of the term of order $(\log \xi^{-1})^{-1}$ in (5.15) would require matching of the expansions in the inner lubrication layer between the two spheres and the 'outer' field determined by the linear ambient flow, for a non-axisymmetric flow field. This appears to be a feasible calculation, but it has not yet been done for the case of two equal spheres. Meanwhile we may estimate the coefficient of $(\log \xi^{-1})^{-1}$ in the expansion for $B$ by fitting the asymptotic form to the values of $B$ obtained from the data of Lin, Lee & Sather (1970) and given in table 1. In figure 3 we show a plot of $B$ and $C'$, $C''$ as a function of $(\log \xi^{-1})^{-1}$. The values of

$B$ and $C'$, $C''$ at $r/a = 2{\cdot}00006$ (which lie outside figure 3 altogether) and $2{\cdot}0001$ given in table 1 are inconsistent with the known limits as $r/a \to 2$ and must be discarded; and the accuracy of the values at $r/a = 2{\cdot}0006$ is suspect. It appears that the relation

$$B(r) = 0{\cdot}4060 - \frac{0{\cdot}78}{\log \xi^{-1}} \tag{5.17}$$

joins the other tabulated values of $B$ smoothly, although the coefficient $-0{\cdot}78$ could prove to be in error by several percent. In paper II we shall regard this relation as the specification of $B$ in the range $2 \leqslant r/a < 2{\cdot}0025$. A similar estimate of the coefficient of $(\log \xi^{-1})^{-1}$ in the asymptotic form for $C'$ and $C''$ in this case of equal spheres is $-0{\cdot}108$.

In his work on the motion of a doublet of two equal spheres in contact immersed in a simple shearing flow, Wakiya (1971) also found the additional rate of dissipation due to the presence of the doublet, at any orientation, by calculating the rate at which work must be done at the surface of a sphere of large radius in order to maintain the flow. For a simple shearing flow in which the ambient velocity has components $(\kappa x_2, 0, 0)$ with respect to a Cartesian co-ordinate system, and for $b/a = 1$, the expression (5.12) for the additional rate of dissipation reduces to

$$\tfrac{20}{3}\pi a^3 \mu \kappa^2 \{1 + K' + (l_1^2 + l_2^2)\, L' + 2 l_1^2 l_2^2\, M'\},$$

where $l_1$, $l_2$, $l_3$ are the components of the unit vector $\mathbf{r}/r$ along the line of centres of the spheres. This is of the same form in $l_1$, $l_2$, $l_3$ as the expression given by Wakiya, and we infer from his numerical coefficients that for equal spheres

$$K' = K'' = -0{\cdot}0472, \quad L' = L'' = 0{\cdot}1928, \quad M' = M'' = 1{\cdot}0508 \tag{5.18}$$

at $r/a = 2$.

It will be seen that these limiting values are consistent with the asymptotic form (4.8) found for a particular combination of the three functions from the exact solution for two equal spheres in an axisymmetric pure straining motion.

### *Calculations for the case $b/a \to 0$*

The only case of different sphere sizes for which similar calculations appear to be feasible at the moment is that in which one sphere is so much larger that its surface can be treated as a plane boundary. Values of the components parallel to the plane surface of the ambient flow force $\mathbf{F}^{(A)}$ and couple $\mathbf{G}^{(A)}$ acting on a stationary sphere of radius $b$ in contact with a plane boundary can be obtained from the calculation by O'Neill (1968) of the force and couple on such a sphere-plane combination in an ambient simple shearing flow. On balancing the components of this force and couple against those due to relative motion of the sphere and plane (as given by (5.2′), (5.2″), (5.3′) and (5.3″)) we can find asymptotic forms of the functions $B$ and $C''$ which are valid for gap widths small compared with $b$. Similarly the asymptotic form of the function $A$ can be found by combining Goren's (1970) calculation of the normal component of $\mathbf{F}^{(A)}$, for a stationary sphere in contact with a plane boundary and at the axis of symmetry of a flow towards the plane, with the estimate (5.2‴) of the normal component of the force $\mathbf{F}^{(S)}$ due to sphere movement.

However, the asymptotic forms of $A$, $B$ and $C''$ obtained in this way are likely to be of limited value in a calculation of the mean stress in a suspension of spheres of non-uniform size, since they are applicable only in a very restricted range of values of the distance between the centres of two spheres of quite different size, viz. $0 \leqslant r-a-b \ll b$, a range which is two orders of magnitude smaller than the radius $a$ of the bigger sphere.

## 6. The relative trajectories of the sphere centres

In all considerations of instantaneous aspects of the interactions of two spheres in a linear ambient flow field, the only consequence of the uniform vorticity at infinity is to contribute a rigid-body rotation with angular velocity $\mathbf{\Omega}$ to the whole flow field, fluid and spheres together. It is for this reason that we are able to find forms for the functions describing instantaneous kinematical and dynamical properties of the interaction which are independent of $\mathbf{\Omega}$. There are, however, some other aspects of the interaction which depend on the history of the motion, one of these being the path traced out by one sphere relative to the other. Somewhat surprisingly, the nature of the family of trajectories of one sphere centre relative to the other is found in paper II to be important in considerations of the probability density function for the separation vector $\mathbf{r}$ in a suspension of spheres subjected to a linear bulk flow, this being a function which occurs in the expression for the bulk stress correct to the order $c^2$. It is important in particular to know whether any of the relative trajectories in a steady linear ambient flow are *closed*, and if so what shape they have. We shall address ourselves here to these questions for two particular steady linear ambient flow fields, viz. pure straining motion and simple shearing motion.

When $E_{ij}$ and $\mathbf{\Omega}$ are independent of time, as we shall assume, the trajectories of one sphere centre relative to another coincide with 'streamlines' of the velocity distribution $\mathbf{V}(\mathbf{r})$ given by (2.2). In the particular case $b/a = 0$, when $A$ and $B$ have the forms (2.12), the relative trajectories coincide with the actual streamlines of the flow due to the sphere $a$ alone in the given ambient flow field.

### *Steady pure straining motion*

When $\mathbf{\Omega} = 0$ and $E_{ij}$ is constant, all the trajectories of one sphere centre relative to another come from infinity and are open, for any value of $b/a$. This is self-evident, although lack of knowledge of the analytical forms of the functions $A$ and $B$ in the expression for the relative velocity $\mathbf{V}$ makes a formal proof difficult. Let $E_1$, $E_2$, $E_3$ be the three principal rates of extension of the bulk flow, where $E_1 > E_2 > E_3$. Then if we introduce spherical polar co-ordinates in $\mathbf{r}$-space, with $\theta = 0$ in the direction of the principal rate of extension $E_1$, and $\theta = \frac{1}{2}\pi, \phi = 0$ in the direction of the principal rate of extension $E_2$, it can be shown from (2.2) that the components of $\mathbf{V}$ are

$$\left.\begin{aligned} V_r &= r(1-A)\{E_1\cos^2\theta + (E_2\cos^2\phi + E_3\sin^2\phi)\sin^2\theta\}, \\ V_\theta &= \tfrac{1}{2}r(1-B)\sin 2\theta\{E_2 - E_1 + (E_3 - E_2)\sin^2\phi\}, \\ V_\phi &= \tfrac{1}{2}r(1-B)(E_3 - E_2)\sin\theta\sin 2\phi. \end{aligned}\right\} \tag{6.1}$$

Reference to table 1 shows that $A \leqslant 1$ and $B < 1$ for all $r$ in the case $b/a = 1$, and the same is true when $b/a \ll 1$. If this is true generally – this is what we are taking to be self-evident – it follows that $V_\theta/\sin 2\theta$ is everywhere negative. Thus the centre of sphere $b$ will move relative to that of sphere $a$ in such a way that $\theta$ tends monotonically to zero if $\theta < \frac{1}{2}\pi$ initially and to $\pi$ if $\theta > \frac{1}{2}\pi$ initially. And at values of $\theta$ near 0 or $\pi$, $V_r > 0$ for all $r$ (and becomes proportional to $r$ ultimately), showing that each trajectory goes to (and comes from) infinity. The form of the trajectories could be calculated explicitly from (6.1) in the cases $b/a = 1$ and $b/a \ll 1$, but will not be needed. The trajectories fill the whole of the accessible part of $\mathbf{r}$-space (that is, the part for which $r \geqslant a+b$), and are qualitatively similar in shape to the streamlines of the flow due to a single rigid sphere in a pure straining motion.

*Steady simple shearing motion*

In this case there is an extensive region of closed trajectories which do not extend to infinity. 'Bound' pairs of spheres in a steady simple shearing motion were observed by Darabaner & Mason (1967). The existence of closed trajectories can also be shown theoretically and their shapes determined. The analysis needed is similar to that used by Cox *et al.* (1968) for the investigation of the closed streamlines in flow due to a single rigid sphere in a steady simple shearing motion.

We choose the ambient flow velocity to have components $(\kappa x_2, 0, 0)$ relative to rectilinear co-ordinates, whence

$$\mathbf{E} = \begin{pmatrix} 0 & \frac{1}{2}\kappa & 0 \\ \frac{1}{2}\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{\Omega} = (0, 0, -\tfrac{1}{2}\kappa).$$

It will be convenient also to employ spherical polar co-ordinates $(r, \theta, \phi)$, with origin at the centre of the sphere with radius $a$ and $\theta = 0$ in the direction of the $x_3$-axis. The corresponding components of the velocity of the centre of sphere $b$ relative to that of sphere $a$ are found from the general expression (2.1) to be

$$\left.\begin{aligned} V_r &= \kappa(1-A)\, r \sin^2\theta \sin\phi\cos\phi, \\ V_\theta &= \kappa(1-B)\, r \sin\theta\cos\theta\sin\phi\cos\phi, \\ V_\phi &= -\kappa r \sin\theta\{\sin^2\phi + \tfrac{1}{2}B(\cos^2\phi - \sin^2\phi)\}. \end{aligned}\right\} \tag{6.2}$$

The path of the second sphere centre relative to the first is then the intersection of the two surfaces given by

$$\frac{1}{r}\frac{\partial r}{\partial\theta} = \frac{V_r}{V_\theta} = \frac{1-A}{1-B}\tan\theta \tag{6.3}$$

and

$$\sin\theta\frac{\partial\phi}{\partial\theta} = \frac{V_\phi}{V_\theta} = \frac{-\sin^2\phi - \frac{1}{2}B(\cos^2\phi - \sin^2\phi)}{(1-B)\cos\theta\sin\phi\cos\phi}. \tag{6.4}$$

From (6.3) we find

$$\frac{1}{r_3}\frac{\partial r_3}{\partial r} = \frac{B-A}{1-A}\frac{1}{r}, \tag{6.5}$$

where $r_3 = r\cos\theta$ is the component of $\mathbf{r}$ in the direction of the $x_3$-axis, and, from (6.4),

$$\frac{\partial(\tan^2\theta\sin^2\phi)}{\partial\theta} = -\frac{B}{1-B}\frac{\sin\theta}{\cos^3\theta},$$

which can also be written as

$$\frac{\partial(r_2^2/r_3^2)}{\partial r} = -\frac{B}{1-A}\frac{r}{r_3^2}, \tag{6.6}$$

where $$r_2 = r\sin\theta\sin\phi.$$

Equation (6.5) can be integrated numerically, once $A$ and $B$ are known as functions of $r$, to give $r_3$ as a function of $r$ as the relation defining one family of surfaces, and then equation (6.6) can be integrated to give $r_2$ as a function of $r$ for the other family of surfaces. The choice of the two constants of integration specifies two particular surfaces, one from each family, and the intersection of these two surfaces specifies a particular trajectory. A convenient way of choosing the two constants of integration specifying a particular trajectory is in terms of the values $R_2$, $R_3$ taken by $r_2$ and $r_3$ at the point (which may not be real) on the trajectory where $r = \infty$. The integrals of (6.5) and (6.6) can then be written as

$$\frac{r_3}{R_3} = \exp\left\{\int_r^\infty \frac{A'-B'}{1-A'}\frac{dr'}{r'}\right\} \tag{6.7}$$

and $$r_2^2 = \frac{r_3^2}{R_3^2}\left\{R_2^2 + \int_r^\infty \frac{B'}{1-A'}\frac{R_3^2}{r_3'^2}r'\,dr'\right\}, \tag{6.8}$$

where $A' \equiv A(r')$, $B' \equiv B(r')$, $r_3' \equiv r_3(r')$.

An open trajectory which extends to infinity is obtained from (6.7) and (6.8) by giving $R_2$ and $R_3$ real values corresponding to the location of the trajectory in the $(x_2, x_3)$-plane far downstream (or upstream). This was done explicitly by Lin *et al.* (1970) for the case of two equal spheres, and some of the open trajectories found by them for $R_3 = 0$ (these being trajectories which lie in the plane of symmetry $x_3 = 0$, with the value of $r_3/R_3$ being given as a function of $r$ by (6.7)) are reproduced in figure 4. For the case $R_2 = 0$, $R_3 = 0$, corresponding to a trajectory which coincides with the $x_1$-axis far downstream, Lin *et al.* found a curve which *asymptotes* to the $x_1$-axis as shown. This suggests the existence of closed trajectories lying between the reference sphere and this limiting open trajectory for which $R_2 = 0$, and one of us (Green 1971) has found them explicitly by extending the calculation of the surfaces (6.7) and (6.8) into the range $R_2^2 < 0$ for this case of equal spheres. Two of these closed trajectories, again with $R_3 = 0$, are shown in figure 4; and the limiting case of the circle $r_1^2 + r_2^2 = 4a^2$ may be shown to correspond to $R_2^2 = -0{\cdot}76a^2$ (approximately).

All trajectories for which $R_2^2 > 0$ are open, and all for which $R_2^2 < 0$ are closed, and the surface bounding the region occupied by closed trajectories is found, by putting $R_2 = 0$ in (6.8) and eliminating $R_3$, to be given by

$$r_2^2 = \exp\left\{2\int_r^\infty \frac{A'-B'}{1-A'}\frac{dr'}{r'}\right\}\int_r^\infty \frac{B'}{1-A'}\exp\left\{-2\int_{r'}^\infty \frac{A''-B''}{1-A''}\frac{dr''}{r''}\right\}r'\,dr'. \tag{6.9}$$

This is an axisymmetric surface which in the case of equal spheres is formed by rotating about the $x_2$-axis the curve in figure 4 corresponding to $R_2 = 0$ together with the mirror-image curve in the lower half of the $(x_1, x_2)$-plane. Somewhat

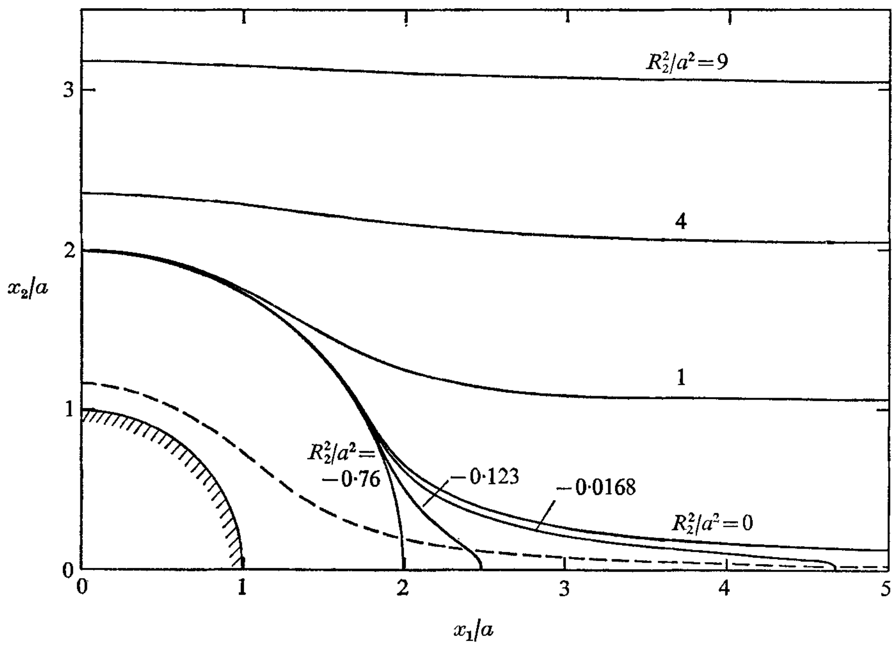


FIGURE 4. The trajectories, in the plane $x_3 = 0$, of the centre of one sphere relative to that of another of the same size in a steady simple shearing motion. The open trajectories ($R_2/a = 1, 2, 3$) and the limiting trajectory ($R_2 = 0$) are taken from Lin *et al.* (1970). Two closed trajectories ($R_2^2/a^2 = -0{\cdot}0168$, $-0{\cdot}123$) have been calculated from (6·7) and (6·8). The boundary of the region of closed streamlines in the steady flow around an isolated sphere is shown as a broken line. All these curves are symmetrical about both the $x_1$ and $x_2$-axes.

surprisingly, the region occupied by closed trajectories has infinite volume. We may see this by noting from (3.6) that, as $r/a \to \infty$,

$$A(r) = O\left(\frac{a^3}{r^3}\right), \quad B(r) \sim \frac{a^5+b^5+\frac{5}{3}a^2b^2(a+b)}{r^5},$$

so that (6.9) becomes

$$r_2^2 \sim \frac{a^5+b^5+\frac{5}{3}a^2b^2(a+b)}{3r^3}. \tag{6.10}$$

The surface which bounds the region of closed streamlines in the case of flow due to a single sphere of radius $a$ in steady simple shearing motion is also given by (6.9), with $b = 0$, and its intersection with the plane $x_3 = 0$ is shown as a broken curve in figure 4; it has a similar asymptotic form, with the coefficient of $r^{-3}$ in (6.10) replaced by $\frac{1}{3}a^5$.

In the case of spheres of equal size it is found that all the closed trajectories lying in the plane of symmetry $x_3 = 0$ pass very close indeed to the inner limit $r/a = 2$ (although this is not so when $b/a \ll 1$, as may be seen from the limiting trajectory for this case in figure 4). It appears that the two equal spheres are swept together by the ambient shearing motion until the gap between their rigid surfaces is small enough for strong 'lubrication' stresses to be generated locally. The inner asymptotic forms for $A$ and $B$ obtained in § 5 are needed in a calculation of the minimum distance between the two surfaces on a closed trajectory which lies near the plane $x_3 = 0$. Significant observations of closed relative trajectories of two spheres of comparable size in a steady simple shearing motion will evidently require the use of spheres whose surfaces are spherical within very narrow tolerances, because a small departure from sphericity might change the trajectory when the separation of the two centres passes through its minimum.

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